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O(3) shift operators and the groups O(4), O(3, 1) and E(3)

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Abstract. Operators shifting the eigenvalues of the O(3) Casimir are constructed from the members of a three-dimensional tensor representation of O(3), and used to give a unified derivation of irreducible representations of the groups O(4), O(3, 1) and the euclidean group E(3).

1. Introduction

In two previous papers (Hughes 1973a, b) the O(3) content of irreducible representations of the group SU(3) was analysed using two pairs of operators which shift the value of l (where $l(l+1)$ is the eigenvalue of the O(3) Casimir L^2) by ± 1 and ± 2. This analysis was rather involved due to the occurrence of l degeneracies, so two simplifications were introduced, the first being the restriction to states of zero m (the eigenvalue of the O(3) generator l_0), and the second being the omission of proofs of previously known results concerning the l content of the representations.

An analogous problem, which is sufficiently easy to avoid having to make such simplifications, is the analysis of irreducible representations of the groups O(4), O(3, 1) and E(3) with respect to their O(3) subgroup. These groups have the common property that their generators, apart from those of O(3), form a three-dimensional tensor representation of O(3), and differ only in the hermiticity conditions and mutual commutation relations of the additional generators. In view of this similarity one should expect the analysis of their irreducible representations to be amenable to a unified treatment, and this can be given using shift operators similar to those used for SU(3). In fact for these groups only one pair of such operators exist, changing l by ± 1, and consequently no l degeneracies can arise. It is this property which makes their treatment so much easier than that of SU(3).

The purpose of this paper is therefore to give this unified treatment and to thereby further illustrate the usefulness of the l shift operators for such problems. No simplifications such as were employed for SU(3) will need to be made, and the m dependence of matrix elements of the shift operators will be explicitly exhibited. Most of the results obtained, apart from those for E(3), will not be new, the case of O(4) being very well known (eg Biedenharn 1961), O(3, 1) having been treated by many authors, notably Naimark (1964). The novel feature of this paper is in the method rather than in the actual results obtained.

When discussing properties common to the groups O(4), O(3, 1) and E(3) we shall denote their Lie algebras by $G(g, h)$, where g and h are parameters depending on, respectively, the hermiticity conditions and mutual commutation relations of the O(3) tensor representation operators. In § 2 we shall write down the shift operators $O_l^{\pm 1}$ for $G(g, h)$ and discuss their hermiticity properties. Expressions will be given for the L^2 commuting products $O_{l\pm 1}^{\mp 1} O_l^{\pm 1}$ in terms of the group invariants, and these used to obtain their matrix elements in terms of g, h and the eigenvalues of the invariants. In § 3 we use the properties of $O_l^{\pm 1}$ to derive the unitary irreducible representations of O(4) and O(3, 1), and analyse them with respect to the O(3) subgroup. A similar analysis is given in § 4 for E(3).

2. The operators $O_l^{\pm 1}$

The groups O(4), O(3, 1) and E(3) are all generated by six operators of which three, l_0 and $l_\pm$, generate the common O(3) subgroup. The remaining generators q_μ, $\mu = 0, \pm 1$, form a three-dimensional irreducible tensor representation of O(3), that is, they have commutators with l_0 and $l_\pm$ of the form

$$[l_0, q_\mu] = \mu q_\mu, \qquad [l_\pm, q_\mu] = \pm(-1)^\mu \{2-\mu(\mu \pm 1)\}^{1/2} q_{\mu\pm 1}. \tag{1}$$

The three groups differ in the hermiticity properties and mutual commutation relations of the q_μ, but since these are not relevant to the construction of the $O_l^{\pm 1}$, we shall not specify them yet.

One may easily verify, using (1), that the operators

$$X = -\tfrac{1}{2}(\sqrt{2} l_0 q_0 - l_+ q_{-1} - l_- q_{+1}) \tag{2}$$

$$Y = hL^2 + \tfrac{1}{2}(q_{+1} q_{-1} + q_{-1} q_{+1} + q_0^2), \tag{3}$$

where h is an arbitrary number and $L^2 = l_+ l_- + l_0(l_0 - 1)$ is the O(3) Casimir, are both O(3) scalars. We therefore label the states upon which our shift operators will operate by the eigenvalues of these operators, together with those of L^2 and l_0. The states will therefore be denoted $|yxlm\rangle$ where y and x will eventually label the irreducible representations of O(4), etc.

The matrix elements of the q_μ can easily be shown (Edmonds 1957) to be given in terms of the reduced matrix elements $(y, x; l \| q \| y', x'; l')$ by the formula

$$\langle y, x; l, m | q_\mu | y', x'; l', m' \rangle = t_\mu (-1)^{l-m} \begin{pmatrix} l & 1 & l' \\ -m & \mu & m' \end{pmatrix} (y, x; l \| q \| y', x'; l') \tag{4}$$

where $\left(\begin{smallmatrix} l & 1 & l' \\ -m & \mu & m' \end{smallmatrix}\right)$ is a 3-j symbol and $t_{+1} = t_0 = -1, t_{-1} = 1$.

We now obtain l shift operators which leave m unchanged and whose action on the states must therefore have the form

$$O_l |y, x; l, m\rangle \propto |y', x'; l, m\rangle.$$

In fact O_l will not change y and x, but this need not be assumed yet. O_l must clearly commute with l_0 and have a commutator with L^2 of the form

$$[L^2, O_l] = 2\lambda O_l. \tag{5}$$

We require O_l to contain the q_μ to only first order; the most general such operator which also commutes with l_0 is

$$O_l = a q_0 + b l_+ q_{-1} + c l_- q_{+1}.$$

The calculation of the values of λ and the corresponding O_l proceeds in an exactly analogous manner to that employed by Hughes (1973a) for the case of SU(3). The values of λ obtained are 0, $l+1$ and $-l$. The corresponding O_l^0 turns out in fact to be the operator X, and $O_l^{\pm 1}$ are given by

$$O_l^{+1} = -\frac{1}{\sqrt{2}}(l^2-l_0^2-1)q_0+\tfrac{1}{2}(l-l_0+1)l_+q_{-1}-\tfrac{1}{2}(l+l_0+1)l_-q_{+1} \tag{6}$$

$$O_l^{-1} = -\frac{1}{\sqrt{2}}(l^2+2l-l_0^2)q_0-\tfrac{1}{2}(l+l_0)l_+q_{-1}+\tfrac{1}{2}(l-l_0)l_-q_{+1}. \tag{7}$$

These last two operators are the same, up to an overall multiplicative constant, as operators constructed by Stone (1956) for the particular case of O(4).

Some properties of the $O_l^{\pm 1}$ should be noted. Firstly, whereas $O_l^{\pm 1}$ shift the l values of kets by ± 1, respectively, they do not act as "pure' shift operators on bras; instead they yield linear combinations of bras of different l values, that is, they are 'right handed' shift operators. Their hermitian conjugates, on the other hand, will be 'left handed' shift operators. Secondly, O_l^{+1} and O_l^{-1} are interchanged by the replacement of l by $-(l+1)$. A detailed discussion of these properties has been given for the shift operators in the case of SU(3) (Hughes 1973a) and this is equally applicable to the operators considered here.

Having shown that the existence of $O_l^{\pm 1}$ does not depend on the hermiticity properties of the q_μ, and in order to proceed further, we impose the conditions $q_0^\dagger = gq_0$ and $q_{+1}^\dagger = gq_{-1}$ where $g = \pm 1$. Taking the hermitian conjugate of (6) yields

$$(O_l^{+1})^\dagger = g\left(-\frac{1}{\sqrt{2}}(l^2+2l-l_0^2+1)q_0-\tfrac{1}{2}(l+l_0+1)l_+q_{-1}+\tfrac{1}{2}(l-l_0+1)l_-q_{+1}\right) \tag{8}$$

showing that $(O_l^{+1})^\dagger \not\propto O_{l+1}^{-1}$. Equations (2) and (3), on the other hand, show that Y is always hermitian whereas $X^\dagger = gX$.

The usefulness of the shift operators relies heavily upon the knowledge of the constant α_l appearing in the equation

$$\langle y', x'; l, m|(O_l^{+1})^\dagger|y, x; l+1, m\rangle = \alpha_l\langle y', x'; l, m|O_{(l+1)}^{-1}|y, x; l+1, m\rangle, \tag{9}$$

and this may be calculated in a manner completely analogous to that used by Hughes (1973a) for the case of SU(3). The result is

$$\alpha_l = \frac{g(2l+1)}{2l+3}. \tag{10}$$

Using (9) and (10) we may relate the matrix elements of the L^2 commuting operators $O_{l\pm 1}^{\mp 1}O_l^{\pm 1}$ to those of $O_l^{\pm 1}$, obtaining

$$\begin{aligned}\langle y, x; l, m|O_{l+1}^{-1}O_l^{+1}|y, x; l, m\rangle \\ &= \frac{1}{\alpha_l}\sum_{y',x'} |\langle y', x'; l+1, m|O_l^{+1}|y, x; l, m\rangle|^2 \\ &= \alpha_l \sum_{y',x'} |\langle y, x; l, m|O_{l+1}^{-1}|y', x'; l+1, m\rangle|^2\end{aligned} \tag{11}$$

$$\langle y, x; l, m|O^{+1}_{l-1}O^{-1}_{l}|y, x; l, m\rangle$$

$$= \alpha_{l-1} \sum_{y',x'} |\langle y', x'; l-1, m|O^{-1}_{l}|y, x; l, m\rangle|^2$$

$$= \frac{1}{\alpha_{l-1}} \sum_{y',x'} |\langle y, x; l, m|O^{+1}_{l-1}|y', x'; l-1, m\rangle|^2. \tag{12}$$

The sums over y' and x' are needed in case the $O^{\pm 1}_{l}$ connect states of differing y and x values. One may also prove, independently of the hermiticity conditions, that

$$\sum_{y,x} \langle y, x; l, m|O^{-1}_{l+1}O^{+1}_{l}|y, x; l, m\rangle = \sum_{y',x'} \langle y', x'; l+1, m|O^{+1}_{l}O^{-1}_{l+1}|y', x'; l+1, m\rangle \tag{13}$$

and finally it can be shown that the $O^{\mp 1}_{l\pm 1}O^{\pm 1}_{l}$ are hermitian operators.

We now close the set l_i, q_μ to form a Lie algebra by assuming mutual commutation relations for the q_μ of the form

$$[q_0, q_{\pm 1}] = \mp\sqrt{2}hl_{\pm}, \qquad [q_{+1}, q_{-1}] = 2hl_0. \tag{14}$$

Any different choice for these commutators which is also consistent with the Jacobi identity can be shown, by replacing the q_μ by suitable linear combinations of the l_i and q_μ, to be equivalent to (14) for some value of h. Also, there are only three basically different cases, depending on whether $h = 0$, $h < 0$, or $h > 0$. However, the case $h < 0$ with $g = \pm 1$ is equivalent to the case $h > 0$ with $g = \mp 1$, so we need consider only $h = 0$ and $h > 0$. Finally, all cases with $h > 0$ can be made equivalent to the case $h = 1$ by a suitable renormalization of the q_μ, so we need consider only $h = 0$ and $h = 1$. There are therefore three essentially different Lie algebras formed from the l_i, q_μ, which we denote by $G(g, h)$. $G(1, 1)$ and $G(-1, 1)$ generate O(4) and O(3, 1), respectively, and $G(1, 0)$ generates E(3). The problem of classifying unitary irreducible representations of these groups will be replaced by the equivalent one of classifying hermitian irreducible representations of the appropriate $G(g, h)$.

It is now easy to show that, providing the h's in (3) and (14) are identified, X and Y commute with the q_μ as well as l_i, and hence also with the $O^{\pm 1}_{l}$. They are therefore the invariants of $G(g, h)$ and their eigenvalues label the irreducible representation. y is always real whereas x is real for $g = 1$ and imaginary for $g = -1$.

Clearly $O^{\mp 1}_{l\pm 1}O^{\pm 1}_{l}$ also commute with X and Y, that is, they commute with all the O(3) scalar operators which may be constructed from the q_μ and l_i, and it is here that the groups generated by $G(g, h)$ differ from SU(3). As a result of this property no l degeneracy arises for $G(g, h)$, for the $O^{\pm 1}_{l\pm 1}O^{\pm 1}_{l}$ act diagonally on $|y, x; l, m\rangle$ so that the result of acting on $O^{\pm 1}_{l}|y, x; l, m\rangle$ with $O_{l}{}^{\mp 1}_{\pm 1}$ is a state proportional to $|y, x; l, m\rangle$. For SU(3) this was not in general the case and the possibility of l degeneracy arose (Hughes 1973a, b).

One may easily show that

$$O^{+1}_{l-1}O^{-1}_{l}|y, x; l, m\rangle = (l^2 - m^2)\{(y+h)l^2 - hl^4 - x^2\}|y, x; l, m\rangle, \tag{15}$$

$$O^{-1}_{l+1}O^{+1}_{l}|y, x; l, m\rangle = ((l+1)^2 - m^2)\{(y+h)(l+1)^2 - h(l+1)^4 - x^2\}|y, x; l, m\rangle. \tag{16}$$

Whereas for the far more complicated case of SU(3) we made the simplification that the shift operators act on states of zero m values, here the m dependence of the eigenvalues of $O^{\mp 1}_{l\pm 1}O^{\pm 1}_{l}$ is shown explicitly and seen, much in the spirit of the Wigner–Eckart theorem, to factor out from the x and y dependent terms. The internal structure of O(3) therefore separates out and so does not complicate the problem of determining the l

content of representations of $G(g, h)$. The (l^2-m^2) term in (15) simply guarantees that the process of obtaining, by successive applications of O_l^{-1} to $|y, x; l, m\rangle$, states of descending l values can never yield states with $l < |m|$.

Equations (11) can now be used to obtain the actions of $O_l^{\pm 1}$ on $|y, x; l, m\rangle$; for instance

$$|\langle y, x; l+1, m|O_l^{+1}|y, x; l, m\rangle|^2 = \alpha_l\langle y, x; l, m|O_{l+1}^{-1}O_l^{+1}|y, x; l, m\rangle.$$

The phases of $|y, x; l, m\rangle$ can be chosen in a self-consistent manner so that

$$\langle y, x; l+1, m|O_l^{+1}|y, x; l, m\rangle$$

is real and positive, and so using (16) we obtain

$$O_l^{+1}|y, x; l, m\rangle = \left(\frac{g(2l+1)\{(l+1)^2-m^2\}\{(y+h)(l+1)^2-h(l+1)^4-x^2\}}{2l+3}\right)^{1/2}|y, x; l+1, m\rangle. \tag{17}$$

Using

$$\begin{aligned}\langle y, x; l+1, m|O_l^{+1}|y, x; l, m\rangle^* &= \langle y, x; l, m|(O_l^{+1})^\dagger|y, x; l+1, m\rangle \\ &= \alpha_l\langle y, x; l, m|O_{l+1}^{-1}|y, x; l+1, m\rangle\end{aligned}$$

then gives

$$O_l^{-1}|y, x; l, m\rangle = g\left(\frac{g(2l+1)(l^2-m^2)\{(y+h)l^2-hl^4-x^2\}}{2l-1}\right)^{1/2}|y, x; l-1, m\rangle. \tag{18}$$

The reduced matrix elements of the q_μ can now be obtained using the matrix elements of X, O_l^{+1} and O_l^{-1}; for instance

$$x = \langle y, x; l, m|X|y, x; l, m\rangle$$

which, on using (2) and (4), can be expressed in terms of $(y, x; l\|q\|y, x; l)$. This yields

$$(y, x; l\|q\|y, x; l) = \left(\frac{2(2l+1)}{l+1}\right)^{1/2}x. \tag{19}$$

In a similar manner (7) and (18), and (6) and (17) yield, respectively,

$$(y, x; l\|q\|y, x; l+1) = -g\left(\frac{2g\{(y+h)(l+1)^2-h(l+1)^4-x^2\}}{l+1}\right)^{1/2} \tag{20}$$

$$(y, x; l\|y, x; l-1) = -\left(\frac{2g\{(y+h)l^2-hl^4-x^2\}}{l}\right)^{1/2}. \tag{21}$$

Finally, using (4), one may calculate all the non-vanishing matrix elements of the q_μ. Thus once the irreducible representations of $G(g, h)$ are known, their analyses with respect to O(3) are given by the results of this section. It remains, therefore, to classify these representations, that is, to obtain the values of y and x and the corresponding range of l, and this will be done in the following two sections.

3. Classification of the irreducible representations

We now consider the classification of hermitian irreducible representations of $G(g, h)$. There are two conditions which must be satisfied, firstly the irreducibility condition which requires the representation space to contain no invariant subspaces, and secondly the hermiticity conditions given in § 2. Also the representations of $G(g, h)$ must reduce to direct sums of irreducible representations of the Lie algebra of O(3) on restriction, which requires the values of x and y to be consistent with l taking on a range of half-integral values.

The irreducibility condition is equivalent to the requirement that any basis vector $|y, x; l, m\rangle$ for the representation space can be obtained from any other basis vector by the successive application of $l_{\pm}$ and $O_l^{\pm 1}$. However, the $l_{\pm}$ act only within irreducible representation spaces of O(3), so since we are interested only in how these subspaces are interconnected we may restrict our considerations to $O_l^{\pm 1}$ which will be used to derive the range of l for given y and x.

Now if for a given irreducible representation of $G(g, h)$ l has an upper bound $\bar{l}$, we must have $O_{\bar{l}}^{+1}|y, x; \bar{l}, m\rangle = 0$. Hence we also have $O_{\bar{l}+1}^{-1}O_{\bar{l}}^{+1}|y, x; \bar{l}, m\rangle = 0$, so $\bar{l}$ must be a zero of the eigenvalue of $O_{\bar{l}+1}^{-1}O_{\bar{l}}^{+1}$. Comparing (16) with (17) one sees that $O_{\bar{l}}^{+1}|y, x; \bar{l}, m\rangle$ vanishes whenever $O_{\bar{l}+1}^{-1}O_{\bar{l}}^{+1}|y, x; \bar{l}, m\rangle$ does, so the above condition is sufficient as well as necessary. By a similar argument the lower bound $\underline{l}$ is a zero of the eigenvalue of $O_{\underline{l}-1}^{+1}O_{\underline{l}}^{-1}$.

The hermiticity conditions are (i) that y and x^2 be real and $x \geqslant 0$ or $\leqslant 0$ according as $g > 0$ or < 0, respectively, and (ii) that the eigenvalues of $O_{l\pm 1}^{\mp 1}O_l^{\pm 1}$ be $\geqslant 0$ or $\leqslant 0$, again according as $g > 0$ or $g < 0$. It may turn out that (ii) is satisfied for all l only if $\bar{l}$ or $\underline{l}$ exist and hence may further restrict the values of y and x so that the eigenvalues of $O_{l\pm 1}^{\mp 1}O_l^{\pm 1}$ vanish for some half-integral value of l. Now the m dependent factors in (15) and (16) are always positive and serve only to guarantee that l never goes below m; hence if we define

$$A = (y+h)(l+1)^2 - h(l+1)^4 - x^2, \qquad B = (y+h)l^2 - hl^4 - x^2 \tag{22}$$

we see that conditions (ii) require that, for all $l \geqslant 0$, A and $B \geqslant 0$ if $g > 0$, and A and $B \leqslant 0$ if $g < 0$.

In the remainder of this section we use these conditions to analyse $G(g, 1)$, returning to $G(1, 0)$ in § 4. Instead of y and x we shall use the variables λ and ν defined by

$$y = \lambda^2 + \nu^2 - 1, \qquad x = \lambda\nu. \tag{23}$$

Equations (22) with $h = 1$ then factorize as follows:

$$A = \{\lambda^2 - (l+1)^2\}\{(l+1)^2 - \nu^2\}, \qquad B = (\lambda^2 - l^2)(l^2 - \nu^2). \tag{24}$$

We observe that A and B are symmetric in λ^2 and ν^2. Also, since y and x^2 are real, λ^2 and ν^2 are either both real or complex conjugates. We now consider the case $g = 1$, corresponding to the group O(4) (and also, of course, O(3) × O(3) and SU(2) × SU(2)), and treat the case $g = -1$ later on in this section.

3.1. Irreducible representations of G(1, 1)

Here $x^2 \geqslant 0$, which is automatically satisfied if λ^2 and ν^2 are complex conjugates; if λ^2 and ν^2 are real, they must have the same sign. We must also have $A \geqslant 0$ and $B \geqslant 0$ for all l, and it is easy to check that neither of these can hold if λ^2 and ν^2 are complex conjugates. When they are real, the above conditions hold providing either $\lambda^2 \geqslant (l+1)^2$ and $l^2 \geqslant \nu^2$ for all l, or $\nu^2 \geqslant (l+1)^2$ and $l^2 \geqslant \lambda^2$ for all l. From the complete symmetry between λ^2 and ν^2, the two possibilities are equivalent so we choose the second as the condition to be satisfied by λ^2 and ν^2. This implies that $\nu^2 > 0$, hence we also have $\lambda^2 \geqslant 0$.

ν and λ are therefore both real; now the only place where their signs enter is in the eigenvalue $x = \lambda\nu$ of X, which may have either of the values $\pm\sqrt{(\lambda^2\nu^2)}$. No loss of generality is therefore lost by requiring $\lambda \geqslant 0$, so that the two values of x will correspond to the choices of sign for ν. If $\lambda = 0$ then also $x = 0$, so in this case the sign of ν is irrelevant and only one case arises.

Since $(l+1)^2 \leqslant \nu^2$ for all l, l must have an upper bound l, which is a zero of $O_{l+1}^{-1}O_l^{+1}$, and therefore of A; hence $l = |\nu|-1$. Also in order that $\lambda^2 \leqslant l^2$ for all l, l must have a lower bound $\underline{l}$ which is a zero of $O_{\underline{l}-1}^{+1}O_{\underline{l}}^{-1}$, and therefore of B; hence $\underline{l} = \lambda$. λ and $|\nu|$ must therefore be half-integral numbers such that $|\nu|-\lambda$ is a positive integer.

To summarize, the irreducible hermitian representations of $G(1, 1)$ are of the type $D^{(\lambda,\nu)}$ where $\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $\nu = \pm(\lambda+1), \pm(\lambda+2), \pm(\lambda+3), \ldots$, and the range of l is $l = \lambda, \lambda+1, \ldots, |\nu|-1$. $D^{(\lambda,\nu)}$ is clearly finite dimensional with dimension $\nu^2-\lambda^2$. $D^{(\lambda,\nu)}$ and $D^{(\lambda,-\nu)}$ are mutually contragredient, whereas $D^{(0,\nu)}$ and $D^{(0,-\nu)}$ are equivalent, that is, $D^{(0,\nu)}$ is self-contragredient. As for the case of SU(3), therefore, contragredient representations contain precisely the same l content.

3.2. Irreducible representations of $G(-1, 1)$

Here $x^2 \leqslant 0$ which precludes ν^2 and λ^2 being complex conjugates, so they must be real and have opposite signs. The symmetry between ν^2 and λ^2 enables us to choose $\lambda^2 \geqslant 0$ and $\nu^2 \leqslant 0$, and taking $\lambda \geqslant 0$ again leads to two cases (except when $\lambda = 0$) depending on the sign of $i\nu$. We must also have, for all l, $A \leqslant 0$ and $B \leqslant 0$, and this leads to various possibilities: (i) $\lambda^2 \geqslant (l+1)^2$ and $\nu^2 \geqslant (l+1)^2$ for all l; this implies that $\nu^2 > 0$ and must be excluded; (ii) $\lambda^2 \leqslant l^2$, $\nu^2 \leqslant l^2$ for all l; the second inequality is automatically satisfied and so l has no upper bound. The first inequality implies that l is bounded below by λ, which must therefore be half integral. We therefore obtain the so-called principal series of irreducible representations, which we denote by $D_{\mathrm{p}}^{(\lambda,\nu)}$, for which $\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, $\mathrm{Re}\,\nu = 0$, $-\infty < \mathrm{Im}\,\nu < \infty$, and l takes on the range of values $l = \lambda, \lambda+1, \lambda+2, \ldots$. $D_{\mathrm{p}}^{(\lambda,\pm\nu)}$ are contragredient representations again, and $D_{\mathrm{p}}^{(0,\nu)}$ is self-contragredient. (iii) We consider the case $\nu = 0$ more carefully. Here the inequalities to be satisfied become $A = \{\lambda^2-(l+1)^2\}(l+1)^2 \leqslant 0$ and $B = (\lambda^2-l^2)l^2 \leqslant 0$. Suppose the minimum value of l is zero; then in order that $A \leqslant 0$ for all l, we must have $\lambda^2 \leqslant 1$. The case $\lambda^2 \leqslant 0$ and $\nu = 0$ is included in the principal series of representations where the roles of λ and ν are interchanged, so we need consider only $0 < \lambda^2 \leqslant 1$. Also, when $l = 0$, although $(\lambda^2-l^2) \geqslant 0$, the factor l^2 guarantees that B still satisfies $B \leqslant 0$. Hence we get hermitian irreducible representations when $\nu = 0$, $0 < \lambda^2 \leqslant 1$ and $\underline{l} = 0$. There are two cases to be considered here: (*a*) $0 < \lambda^2 < 1$. In this case A never vanishes and so l has no upper bound. We obtain in this case the so-called supplementary series of representations, $D_{\mathrm{s}}^{(\lambda,0)}$, for which $\nu = 0$, $\mathrm{Im}\,\lambda = 0$ and $0 < \mathrm{Re}\,\lambda < 1$, and $l = 0, 1, 2, \ldots$. $D_{\mathrm{s}}^{(\lambda,0)}$ is self-contragredient. (*b*) If $\lambda^2 = 1$ and $\nu = 0$ then $A \leqslant 0$ and $B \leqslant 0$ are satisfied either if l is bounded below by λ and has no upper bound—this case is already included in the principal series of representations—or if λ is bounded below by 0. In this case $l = 0$ is a zero of both A and B so that l is also bounded above by 0. This case gives the trivial representation $D_{\mathrm{T}}^{(1,0)}$ for which $l = 1$, $\nu = 0$ and $l = 0$.

The above cases exhaust all possible hermitian irreducible representations of $G(-1, 1)$ and yield, on exponentiation, all irreducible unitary representations of O(3, 1) and its cover group SL(2, C). The results obtained are completely equivalent to those obtained, for instance, by Naimark (1964).

4. Irreducible representations of $G(1, 0)$

We finally classify the hermitian irreducible representations of $G(1, 0)$, which is the (non-semi-simple) Lie algebra of the group E(3) of rotations and translations in three dimensions. Since $h = 0$,

$$A = y(l+1)^2 - x^2, \qquad B = yl^2 - x^2. \tag{25}$$

Since $g = 1$, x and y are both real and A and B are non-negative. Writing $y = p^2$, the conditions which must be satisfied become $p^2(l+1)^2 - x^2 \geqslant 0$ and $p^2l^2 - x^2 \geqslant 0$ for all l. Since $x^2 \geqslant 0$ and $(l+1)^2 > 0$, the first inequality implies that $p^2 \geqslant 0$. We have two possible cases: (i) $p^2 > 0$. In this case both the above inequalities are satisfied if $l^2 \geqslant x^2/p^2$ for all l. The sign of p is irrelevant and we choose $p > 0$, so that $l \geqslant |x|/p$ for all l. $|x|/p$ must therefore be the minimum value of l, and is therefore half integral. Clearly l has no upper bound, so the irreducible representation is infinite dimensional. We therefore obtain the representations $D^{(p,x)}$ for which $\mathrm{Im}\, p = 0$, $0 < \mathrm{Re}\, p < \infty$, and $x = \pm p(0, \frac{1}{2}, 1, \frac{3}{2}, \ldots)$, and l takes on the range of values $l = |x|/p$, $|x|/(p+1)$, $|x|/(p+2), \ldots$; $D^{(p,\pm x)}$ are clearly contragredient representations. (ii) $p = 0$. Here A, $B \geqslant 0$ can be satisfied only if $x^2 = 0$, so that in fact A and B are both zero. This implies not only that l possesses an upper and a lower bound, but also that these bounds are equal. Hence only one value of l occurs, which may be either integral or half integral. These representations are all self-contragredient and finite dimensional. (Note that finite dimensional hermitian irreducible representations of non-compact Lie algebras are not excluded provided the Lie algebra is not semi-simple, so no contradiction occurs here.) Using (19)–(21) we see that for these representations all matrix elements of the q_μ vanish; they are therefore just the irreducible representations of the Lie algebra of O(3).

5. Conclusions

We summarize the advantages of the methods used in this paper over alternative approaches. (i) Three classes of groups can be analysed by a single unified method; (ii) no specific realizations of the representation spaces were employed; (iii) the analysis of representations of O(3, 1) was obtained rather more directly than by methods not employing the shift operators $O_l^{\pm 1}$; (iv) finally, the existence of the $O_l^{\pm 1}$ was shown to be independent of the hermiticity properties and mutual commutation relations of the q_μ.

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